MATH 2010B Advanced Calculus I, Lecture Notes Week 5

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Last time ...

$$f(x_1, \dots, x_n) : \mathbb{R}^n \longrightarrow \mathbb{R} \quad \text{multivariable functions}$$

- level set ("slicing")

Limit of multivariable functions

Def: $f : \mathbb{R}^n \to \mathbb{R}$ multivariable function

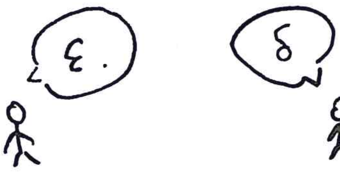
"$\lim\limits_{\vec{x} \to \vec{x}_0} f(\vec{x}) = L$" $\Longleftrightarrow$ $\varepsilon - \delta$ definition

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

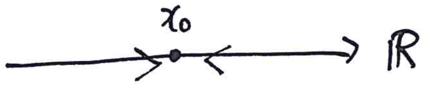
if $0 < \| \vec{x} - \vec{x}_0 \| < \delta$

then $| f(\vec{x}) - L | < \varepsilon$

Dynamic picture



1-dim:

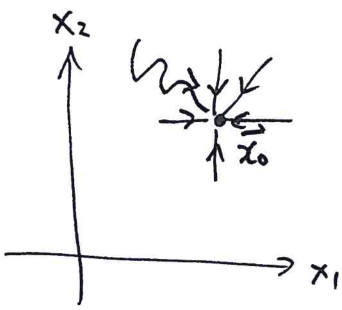


$$\lim_{x \to x_0^-} = \lim_{x \to x_0^+} = L$$

$$\Downarrow$$

$$\lim_{x \to x_0} = L$$

2-dim:



Note: There are many ways to approach $\vec{x}_0$ in dim ≥ 2.

Remark: $n \geq 2$.

$\lim\limits_{\vec{x} \to \vec{x}_0} f(\vec{x}) = L \Longleftrightarrow$ limit of $f(x)$ approach L along any possible path.

Useful to show that limit does not exist.

Examples

1) $$\lim_{(x,y)\to(0,0)} (x^2+y^2) = 0$$

<u>Pf</u>: Fix an $\varepsilon > 0$,

<u>Goal</u>: find a $\delta > 0$ s.t.

$$\underbrace{\|(x,y)-(0,0)\|}_{\sqrt{x^2+y^2}} < \delta \Rightarrow |\underbrace{x^2+y^2-0}_{x^2+y^2}| < \varepsilon$$

If I take $\delta^2 \le \varepsilon$ (say $\delta = \sqrt{\varepsilon}$)

then <u>Goal</u> is met. ■

<u>Observe</u>:

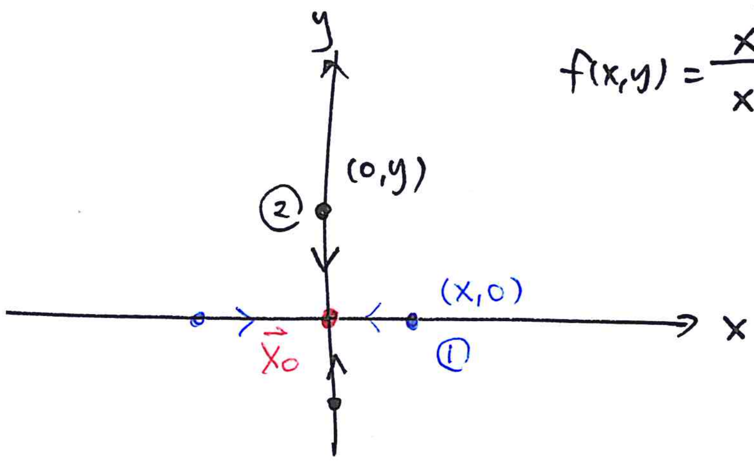
$$\left.\begin{array}{l} \lim_{x\to 0}\lim_{y\to 0}(x^2+y^2) = \lim_{x\to 0} x^2 = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \lim_{y\to 0}\lim_{x\to 0}(x^2+y^2) = \lim_{y\to 0} y^2 = 0 \end{array}\right\} \text{double limit}$$

2) $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = ?$$ does not exist.

<u>Observe</u>: ① $$\lim_{x\to 0}\lim_{y\to 0}\frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0}\frac{x^2}{x^2} = 1$$

$$\neq \quad \Rightarrow \text{2D limit does not exist.}$$

② $$\lim_{y\to 0}\lim_{x\to 0}\frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0}\frac{-y^2}{y^2} = -1$$



$$f(x,y) = \frac{x^2-y^2}{x^2+y^2}$$

3) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = ?$ does not exist.

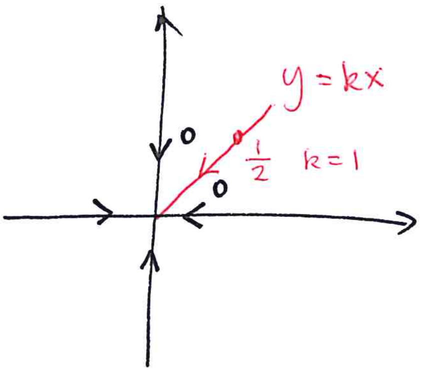
Observe:

$$\lim_{x\to 0}\lim_{y\to 0}\frac{xy}{x^2+y^2} = \lim_{x\to 0}\frac{0}{x^2} = 0$$

$$\|$$

$$\lim_{y\to 0}\lim_{x\to 0}\frac{xy}{x^2+y^2} = \lim_{y\to 0}\frac{0}{y^2} = 0$$

If $y = kx$,



$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} \frac{xy}{x^2+y^2} = \lim_{x\to 0}\frac{kx^2}{x^2+k^2x^2}$$

$$= \lim_{x\to 0}\frac{k}{1+k^2}$$

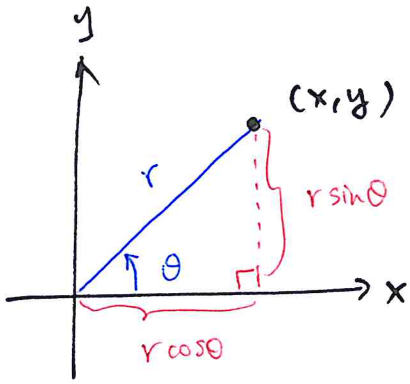
$$= \frac{k}{1+k^2}$$

eg. $k=1$

$$= \frac{1}{2} \neq 0$$

Remark: limit exists & equal along every line $y=kx$ $\not\Rightarrow$ 2D limit exists.

Polar Coordinates (in $\mathbb{R}^2$)



polar coord. (r,θ) $\longleftrightarrow$ rectangular coord. (x,y)

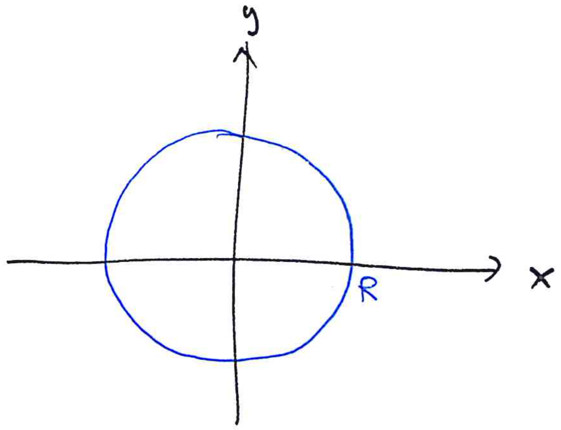
$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$$\boxed{\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}}$$

← only defined up to $2n\pi$.

Polar representation

(1)



($R>0$)

Circle: $x^2+y^2=R^2$ rectangular.

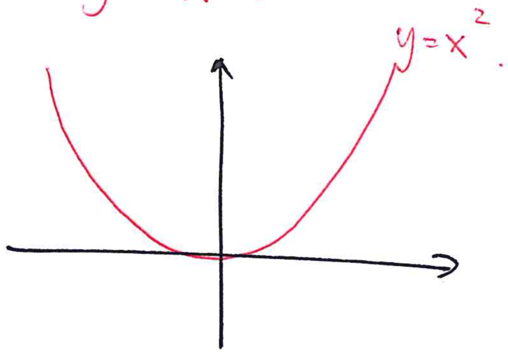
$\boxed{r=R}$ polar coord.

Systematic way:

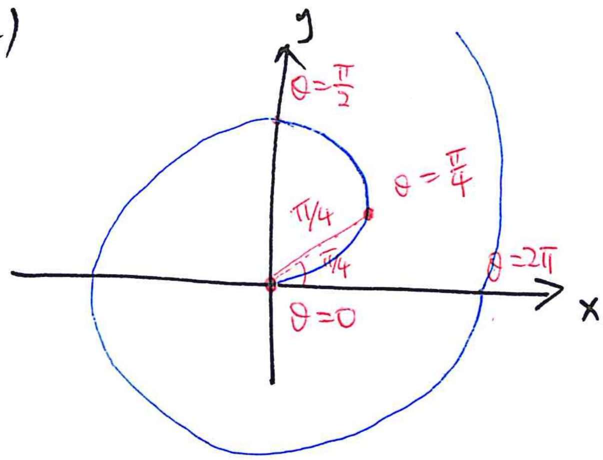
Sub. $x=r\cos\theta$, $y=r\sin\theta$ into the rectangular eq^{n}:

$$x^2+y^2=R^2$$
$$(r\cos\theta)^2+(r\sin\theta)^2=R^2$$
$$r^2(\cos^2\theta+\sin^2\theta)=R^2$$
$$r^2=R^2$$
$$r=R$$

Q: Write the polar eq^{n} for $y=x^2$.



(2)



Polar eq^{n}: $\boxed{r=\theta}$

"Archemedean spiral"

Q: Sketch the polar curve

$\boxed{r=1-a\cos\theta}$

(i) $0<a<1$

(ii) $a=1$

(iii) $a>1$. (tricky)